

# Kinetic energy cascades in quasi-geostrophic convection in a spherical shell

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**Abstract.** We consider triadic nonlinear interaction in the Navier-Stokes equation for quasi-geostrophic convection in a spherical shell. This approach helps understanding the origin of kinetic energy transport in the system and the particular scheme of mode interaction, as well as the locality of the energy transfer. The peculiarity of convection in the sphere, concerned with excitation of Rossby waves, is considered. The obtained results are compared with our previous study in Cartesian geometry.

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## 1. Introduction

The redistribution of the physical fields is a quite widespread phenomenon in Nature. If we consider convective systems with dissipation, there are then two major effects which work in the opposite directions: diffusion (may be turbulent), which leads to the homogenization of the fields, and nonlinear interactions, which can produce large gradients and, in fact, cause the self-organization of the system. The most impressive examples of such behavior is the appearance of coherent structures in convective systems [1], topological pumping [2] and turbulent diamagnetism in the dynamo [3].

The considered effects demonstrate pumping of the fields in physical space to some domain of a scale smaller than the main scale of the system. If the subdomain is quite small, then it is clear that pumping should also be accompanied by the redistribution of the fields in wave space, e.g., the energy fluxes between the different scales relevant to the nonlinear process,  $\alpha$ -effect and separation in the scales of magnetic helicity [4, 5] in the mean-field dynamo theory [6], already mentioned coherent structures in convection. It is worth noting that fluxes exist in the wave space even in quasistationary states provided that the scales of the energy injection into the system and the energy dissipation are different, see the introduction to the problem in [7] and the review of some recent results in [8].

In general, such fluxes can be defined for various physical quantities, and the direction of their propagation is prescribed by many factors: by the nature of the considered quantity, the number of dimensions in the system and even the geometry of the system (as we shall see later). Restricting our further research to convection, we recall that for the 3D homogeneous isotropic turbulence (in the absence of rotation) there is a direct energy cascade of the kinetic energy  $E_K$  from the large scales to the small scales, where dissipation takes place. The second invariant of the Navier-Stokes equation in 3D in the inviscid limit is kinetic helicity  $\chi$ . However, for isotropic turbulence its mean value is zero and its mean flux is also zero.

The situation changes in 2D, where the cascade of kinetic energy is inverse: from the small to the large scales. Two-dimensional idealization was a useful approach for describing geophysical turbulence and was able to capture many important features of the flow. With regard to the more realistic models, one should consider quasi-geostrophic flows, which in view of their properties are somewhere inbetween 3D and 2D. In such a flow, one still has 3 dimensions, but the dependence of angular rotation ( $z$  coordinate) on direction is degenerated, see [9]. This flow is known by its structures elongated along  $z$ . The perpendicular scale of these structures is very small and defined by the value of the Ekman number  $E$ . At the critical Rayleigh number, the first columnar mode has horizontal and vertical scales  $\mathcal{O}(E^{1/3})$ ,  $\mathcal{O}(1)$ , respectively [10]. Note that  $E \sim 10^{-15}$  in the Earth's liquid core, and the horizontal scale is extremely small. As follows from our previous study for the rotating rectangular box heated from below with periodic boundary conditions in the horizontal plane [11], these structures can supply kinetic energy in both directions: to the larger as well as to the smaller scales, which

can be important for understanding the energy budget in the planetary cores and for constructing semi-empirical models of turbulence. However, the solution of the same equations in plane geometry differs from that in the sphere.

In Cartesian geometry, the increase of the heat sources characterized by the Rayleigh number leads to the gradual increase of the number of growing wave modes with smaller and larger scales than the scale of the first mode. However, in spherical geometry the increase of Ra excites Rossby waves. As a result the columns start to oscillate in the direction of the axis of rotation of the sphere, which leads to the rotation of columns around it. The direction of this rotation is defined by the slope of the outer boundary. For a spherical shell it is prograde and for a concave surface it has the opposite direction [12]. The appearance of this rotation (even differential) corresponds to the axi-symmetrical mode in spectral space. Thus the scenarios of energy transfer in the plane and in spherical geometries can be different. This is the motivation of our present study. Here, following [11], we discuss the fluxes of kinetic energy in the standard Boussinesq equations used in the geodynamo for the different magnitudes of Ra and compare the results with the Cartesian geometry survey.

## 2. Equations

The thermal convection process driven by the flows of incompressible fluid ( $\nabla \cdot \mathbf{V} = 0$ ) in the Boussinesq approximation in a spherical shell ( $r_{\text{ICB}} \leq r \leq r_{\text{CMB}}$ ) rotating with the angular velocity  $\Omega$  in the  $z$ -direction is described by the Navier-Stokes equation

$$\text{Pr}^{-1} \mathbf{E} \left( \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) = -\nabla P + \mathbf{F} + \mathbf{E} \nabla^2 \mathbf{V} \quad (1)$$

and the heat flux equation for temperature fluctuations  $T$

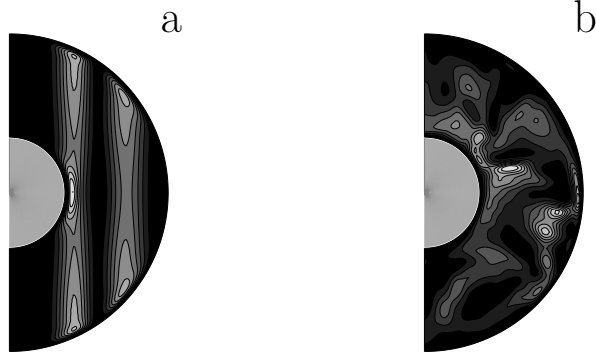
$$\frac{\partial T}{\partial t} + (\mathbf{V} \cdot \nabla) (T + T_o) = \nabla^2 T, \quad (2)$$

where  $T_o = \frac{r_{\text{ICB}}/r - r_{\text{CMB}}}{r_{\text{CMB}} - r_{\text{ICB}}}$  is the solution of the heat flux equation with fixed temperatures (1, 0) at the boundaries ( $r_{\text{ICB}}, r_{\text{CMB}}$ ) in the absence of convection. Hereinafter ICB denotes the inner core boundary, CMB the core mantle boundary and  $(r, \theta, \varphi)$  is the spherical system of coordinates.

The equations are scaled with the outer radius of the shell  $L$ , which makes the dimensionless radius  $r_{\text{CMB}} = 1$ ; the inner core radius  $r_{\text{ICB}}$  is equal to 0.35, which is the value valid for the Earth. Velocity  $\mathbf{V}$ , pressure  $P$  and the typical diffusion time  $t$  are measured in units of  $\kappa/L$ ,  $\rho\kappa^2/L^2$  and  $L^2/\kappa$ , respectively, where  $\kappa$  is the thermal diffusivity,  $\rho$  is density,  $\text{Pr} = \frac{\nu}{\kappa}$  is the Prandtl number,  $\mathbf{E} = \frac{\nu}{2\Omega L^2}$  is the Ekman number and  $\nu$  is the kinematic viscosity.

Force  $\mathbf{F}$  includes the Coriolis and Archimedean effects:

$$\mathbf{F} = -\mathbf{1}_z \times \mathbf{V} + \text{Ra} T r \mathbf{1}_r, \quad (3)$$



**Figure 1.** Meridional section of kinetic energy  $E_K$  for a)  $E = 2 \cdot 10^{-4}$ ,  $Pr = 1$ ,  $Ra = 40$ ,  $Re \sim 17$ ,  $(0, 60)$ , with rotation, and b)  $E = 1$ ,  $Pr = 1$ ,  $Ra = 2.5 \cdot 10^6$ ,  $Re \sim 3.4 \cdot 10^2$ ,  $(0, 5 \cdot 10^4)$  without rotation (Coriolis force is switched off). Numbers in parentheses correspond to the range of the field.

where  $\mathbf{1}_z$  is the unit vector along the axis of rotation,  $Ra = \frac{\alpha g_o \delta T L}{2 \Omega \kappa}$  is the modified Rayleigh number,  $\alpha$  is the coefficient of volume expansion,  $\delta T$  is the unit of temperature, and  $g_o$  is the gravitational acceleration at  $r = r_{CMB}$ .

The inner core,  $r \leq r_{ICB}$ , with surface  $S_{ICB}$ , can rotate around axis  $z$  due to the viscous torque, caused by the no-slip boundary conditions, used at both the boundaries  $r_{ICB}$ ,  $r_{CMB}$ . The momentum equation for the angular velocity  $\omega$  of the inner core has the form:

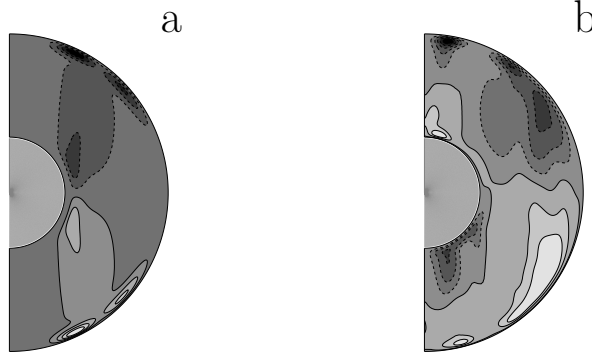
$$Pr^{-1} I \frac{\partial \omega}{\partial t} = r_{ICB} \oint_{S_{ICB}} \frac{\partial V_\varphi}{\partial r} \sin \theta dS, \quad (4)$$

where  $I$  is the inner-core moment of inertia along the  $z$ -axis.

Equations (1–4) were solved using the standard spherical functions decomposition accompanied by a poloidal-toroidal decomposition of vector field  $\mathbf{V}$ . The fast Chebyshev transform was used in the  $r$ -direction. The mesh grid in physical space was  $128^3$ . The Fortran code was parallelized in the  $r$ -direction using MPI. The details of the spherical function and Chebyshev polynomial decomposition can be found in [13, 14, 15].

### 3. Quasi-geostrophic convection

Rapid rotation is a quite familiar phenomenon in geophysics. A strong Coriolis force leads to the appearance of elongated structures (columns) along the axis of rotation, in contrast to the cellular patterns in the non-rotating regime, where the spherically symmetrical (when the fields are averaged in time) buoyancy forces dominate, see Fig. 1. These columns with a horizontal scale  $l_d \sim E^{1/3} L \ll 1$  rotate around their axes, so that the net helicity in the northern hemisphere for small Rayleigh numbers is negative and in the southern hemisphere positive, see Fig. 2(a). Usually, the combination of the mean

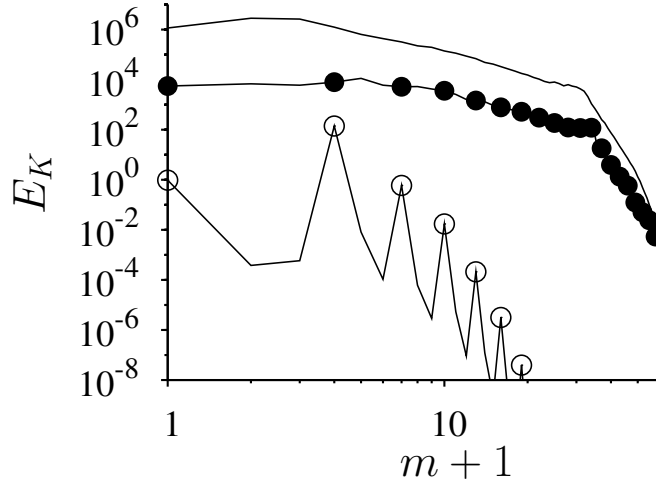


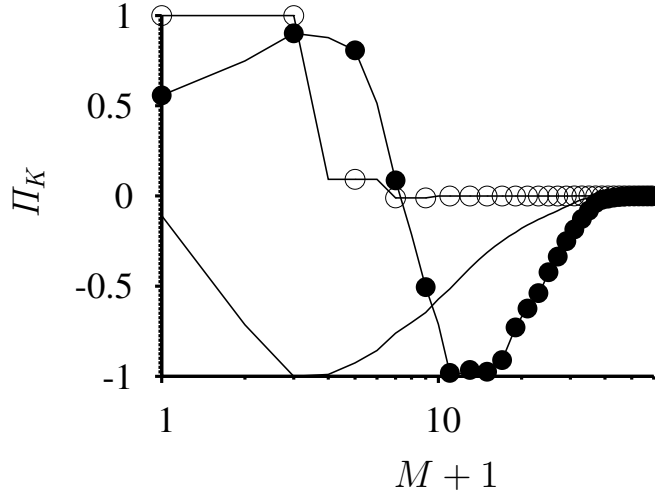
**Figure 2.** Meridional section of the mean kinetic helicity  $\chi$  for  $E = 2 \cdot 10^{-4}$ ,  $Pr = 1$  a)  $Ra = 40$ ,  $Re \sim 17$ ,  $(-1.2 \cdot 10^3, 1.2 \cdot 10^3)$  and b)  $Ra = 4 \cdot 10^2$ ,  $Re \sim 2 \cdot 10^2$ ,  $(-3 \cdot 10^5, 3 \cdot 10^5)$ . The dotted isolines correspond to negative values.

kinetic helicity  $\chi = \langle \mathbf{V} \cdot \text{rot } \mathbf{V} \rangle$  (closely connected with the  $\alpha$ -effect) with differential rotation is considered as an explanation of the existence of the large-scale planetary magnetic fields. The helicity depends on the regularity of the flow and ratio of the Coriolis and Archimedean forces: the increase of the heat sources leads to the spread of convection to the Taylor cylinder (TC) accompanied by strong differential rotation. As a result,  $\chi$  changes sign in the middle of the spherical shell in TC, being positive at the inner-core boundary (ICB) and negative at the core-mantle boundary (CMB) in the northern hemisphere (the helicity in TC still retains its dipole structure with respect to the equatorial plane). Before we refer the reader to the more detailed study of kinetic helicity, including the influence of the magnetic field on  $\chi$  [16], let us stress that, from the point of view of the mean-field theory [6],  $\chi$  in TC and outside it are not the same: in TC  $\chi$  is produced by the large-scale motions near the boundaries, and in the outer part it has a cyclonic nature with  $l_d \ll 1$ , so that the separation of the fields into the large and small scales, adopted in the theory, is valid.

#### 4. Spectral properties

As follows from the behavior of convection in physical space, rotation leads to the appearance of the small scales in the horizontal plane. Hereinafter we consider the spectral properties in the azimuthal direction, bearing in mind that the one-dimensional spectrum  $S(m)$  of field  $F(r, \theta, \varphi)$  means:  $S(m) = \int \int f(m) \overline{f(m)} r^2 \sin \theta dr d\theta$ , where  $f$  is the Fourier transform of  $F$  and  $\overline{f}$  is the complex conjugate of  $f$ . The spectra of three regimes are presented in Fig. 3.





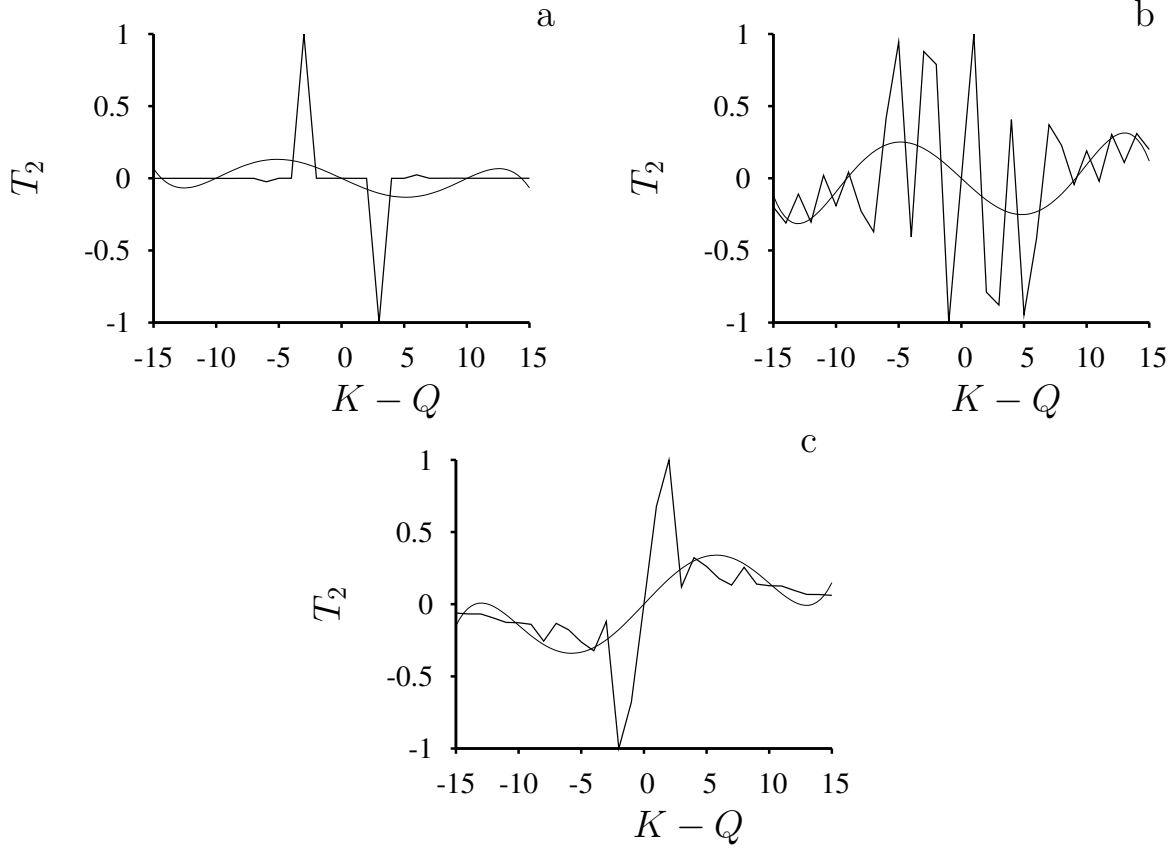
**Figure 4.** Integral flux  $\Pi_K$  of kinetic energy as a function of the azimuthal wavenumber for the three regimes with the same labels as in Fig.3.

non-rotating regime,  $\Pi$  is always negative, which means that the large scales feed the small scales (the direct cascade of energy). For the small  $M$ , the slope of increasing  $\Pi$  is constant, which means that the flux to the small scales is non-local. The same situation holds for the larger  $M$ , where  $\Pi$  increases, but the slope also remains constant. Due to this non-locality our solution differs from that for the Cartesian geometry in [11, 17].

For the geostrophic convection with small  $Ra$  we observe the inverse cascade of energy for  $M = 0 - 2$ ,  $\Pi > 0$ . The break of the curve at  $M = 3$  means that, due to the sharp decrease of the spectrum, approximately the whole kinetic energy is in the sphere and  $\Pi$  tends to zero. This is the reason why the direct cascade cannot be seen in the plot for large  $M$ . The increase of  $Ra$  decreases the relative flux to the axi-symmetrical mode twice. For larger  $M$ ,  $\Pi$  changes sign, which corresponds to the transition from the inverse cascade regime for  $M < 3$  to the direct cascade of the kinetic energy for  $M > 3$ .

The rotation changes not only the direction of energy transfer over the spectrum, but also the structure of the triangle in wave space, when two modes with  $m = P$  and  $m = Q$  produce the third mode  $m = K$ . To check this possibility, we have constructed the  $T_2$  flux function which describes the input of energy of the  $P$ -mode to the  $K$ -mode:  $T_2(P, K) = -[(\mathbf{V}(P) \cdot \nabla) \mathbf{V}] \cdot \mathbf{V}(K)$  ‡. It is convenient to present  $T_2$  as a function of  $K - P$ , see Fig. 5. There is clear evidence of the non-local transfer of energy between the modes for small  $Ra$  under rotating convection. The shift of the extrema from the zero point describes the non-locality of interactions. This phenomenon can be explained as follows: the extrema correspond to the period ( $m = 3$ ) between the local maxima in the spectra, see Fig. 3. Negative  $T_2$  for positive arguments means an inverse cascade. The increase of  $Ra$  leads to the complex interaction between the modes with both the

‡ By definition  $T_2$  is antisymmetric function.



**Figure 5.** The fluxes of kinetic energy  $T_2(K - P)$  for the three regimes, see Fig.3,4. Thin line corresponds to the 5<sup>th</sup> order polynomial approximation.

direct and inverse cascades with a different level of non-locality. The approximation of the curve with polynomials demonstrates the evident inverse cascade and non-locality in interactions, see Fig. 5. The comparison of the non-local characteristics for the spherical and Cartesian geometries demonstrates the increase of the non-locality transfer in the former case. There are many reasons for this phenomenon. Due to the spherical boundary, the spectra on  $m$  accumulate various scales in the  $z$ -directions. The pure statistics in the spherical problem may be more important: if the number of the columns is  $\approx k_\perp^2$  in the flat geometry, their number is much smaller in the spherical geometry, because they are mainly distributed near TC. The other reason is that considered in [17]:  $k_\perp = 8$  for the columns is larger than that in the spherical geometry ( $M = 3$ ) and we stand far away from the similarity region, even if we have similar resolutions in the models. That is why Cartesian geometry is so often used in MHD simulations to get scalings.

## 5. Discussion

We have tried to present the regimes of geostrophic convection, well-known in the geodynamo community, by traditional tools of the turbulent community. Even if the



solution is quasi-stationary in physical space, there are non-zero fluxes of energy in wave space, proving that convection is not localized in wave space. Our results demonstrate that recent simulations for the usually used grids do not exhibit local transfer of energy, even without rotation. Rotation brings inverse flux into the first harmonics. In this way we can consider the excitation of the axi-symmetrical rotation of the columns around geographical axes to be the result of the inverse cascade. This phenomenon has no analog in the case of convection in the box, where rotation destroys rotating rolls with  $m = 0$ , for details of the Küppers-Lortz instability refer to [18, 19]. In some sense, the periodical boundary condition used in the Cartesian geometry corresponds to the case with the zero curvature of the boundary considered in [12].

As can be seen from the presented simulations, the relative location of energy transfer will increase with increasing resolution of the model, but we believe that the mechanism of excitation of the axi-symmetrical flow due to the non-local inverse energy transfer will be the same. We hope that these results will also be interesting for solar modeling, where this technique could be used to check the energy exchange between the different radial layers and could thus help in constructing more realistic turbulent models.

## References

- [1] Tabeling P 2002 *Phys. Reports.* **362** 1
- [2] Drobyshevski E M, Yuferev B S 1974 *J. Fluid. Mech.* **65** 33
- [3] Zeldovich Ya B, Ruzmaikin A A, Sokoloff D D 1983 *Magnetic fields in astrophysics* (NY, Gordon and Breach)
- [4] Brandenburg A, Subramanian K 2005 *Phys. Rep.* **417** 1
- [5] Hejda P, Reshetnyak M, 2010 *Geophys. Astrophys. Fluid Dynam.* **104** 6 25
- [6] Krause F, Rädler K-H 1980 *Mean field magnetohydrodynamics and dynamo theory* (Berlin, Akademie-Verlag)
- [7] Frisch U 1995 *Turbulence: the legacy of A N Kolmogorov* (Cambridge, Cambridge University Press)
- [8] Verma M 2004 *Phys. Reports.* **401** 229
- [9] Pedlosky J 1987 *Geophysical Fluid Dynamics* (NY, Springer-Verlag)
- [10] Busse F H 1970 *J. Fluid Mech.* **44** 441
- [11] Reshetnyak M, Hejda P 2008 *Nonlin. Processes Geophys.* **15** 873
- [12] Busse F H 2002. *Phys. Fluids.* **14** 4 1301
- [13] Glatzmaier G A 1984 *J. Comp. Physics.* **55** 461
- [14] Tilgner A 1999 *Int. J. Numer. Meth. Fluids.* **30** 713
- [15] Simitev R 2004 *Ph.D. Thesis: Convection and Magnetic Field Generation in Rotating Spherical Fluid Shells* (Bayreuth, University of Bayreuth)
- [16] Sreenivasan B, Davidson P A 2008 *Phys. Fluids.* **20** 085104-1
- [17] Hejda P, Reshetnyak M 2009 *Phys. Earth Planet. Int.* **177** 152
- [18] Küppers G, Lortz D 1969 *J. Fluid Mech.* **35** 609
- [19] Jones C A, Roberts P H 2000 *J. Fluid Mech.* 404 311